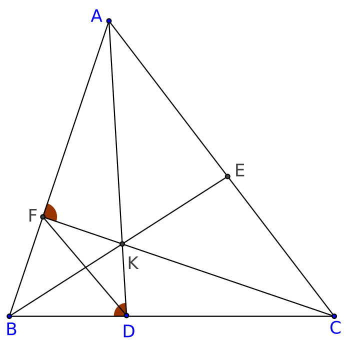
Problems and Solutions: CRMO-2011

1. Let ABC be a triangle. Let D, E, F be points respectively on the segments BC, CA, AB such that AD, BE, CF concur at the point K. Suppose $BD/DC = BF/FA$ and $\angle ADB = \angle AFC$. Prove that $\angle ABE = \angle CAD$.



Solution: Since $BD/DC = BF/FA$, the lines DF and CA are parallel. We also have $\angle BDK = \angle ADB = \angle AFC = 180° - \angle BFK$, so that $BDKF$ is a cyclic quadrilateral. Hence $\angle FBK = \angle FDK$. Finally, we get

$$\begin{aligned}\angle ABE = \angle FBK &= \angle FDK \\ &= \angle FDA = \angle DAC,\end{aligned}$$

since $FD \parallel AC$.

2. Let $(a_1, a_2, a_3, \ldots, a_{2011})$ be a permutation (that is a rearrangement) of the numbers $1, 2, 3, \ldots, 2011$. Show that there exist two numbers j, k such that $1 \le j < k \le 2011$ and $|a_j - j| = |a_k - k|$.

Solution: Observe that $\sum_{j=1}^{2011} (a_j - j) = 0$ since $(a_1, a_2, a_3, \ldots, a_{2011})$ is a permutation of $1, 2, 3, \ldots, 2011$. Hence $\sum_{j=1}^{2011} |a_j - j|$ is even. Suppose $|a_j - j| \ne |a_k - k|$ for all $j \ne k$. This means the collection $\{|a_j - j| \ : \ 1 \le j \le 2011\}$ is the same as the collection $\{0, 1, 2, \ldots, 2010\}$ as the maximum difference is 2011-1=2010. Hence

$$\sum_{j=1}^{2011} |a_j - j| = 1 + 2 + 3 + \cdots + 2010 = \frac{2010 \times 2011}{2} = 2011 \times 1005,$$

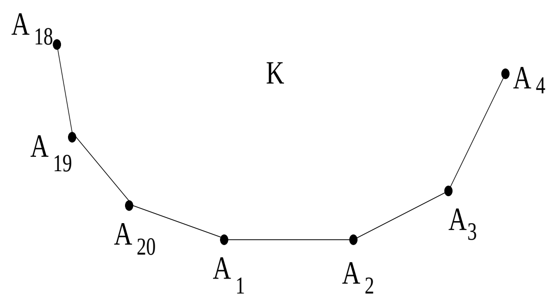
which is odd. This shows that $|a_j - j| = |a_k - k|$ for some $j \ne k$.

3. A natural number n is chosen strictly between two consecutive perfect squares. The smaller of these two squares is obtained by subtracting k from n and the larger one is obtained by adding l to n. Prove that $n - kl$ is a perfect square.

Solution: Let u be a natural number such that $u^2 < n < (u+1)^2$. Then $n - k = u^2$ and $n + l = (u+1)^2$. Thus

$$\begin{aligned} n - kl &= n - \big(n - u^2\big)\big((u+1)^2 - n\big) \\ &= n - n(u+1)^2 + n^2 + u^2(u+1)^2 - nu^2 \\ &= n^2 + n\Big(1 - (u+1)^2 - u^2\Big) + u^2(u+1)^2 \\ &= n^2 + n\big(1 - 2u^2 - 2u - 1\big) + u^2(u+1)^2 \\ &= n^2 - 2nu(u+1) + \big(u(u+1)\big)^2 \\ &= \big(n - u(u+1)\big)^2. \end{aligned}$$

4. Consider a 20-sided convex polygon K, with vertices $A_1, A_2, \ldots, A_{20}$ in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not.)



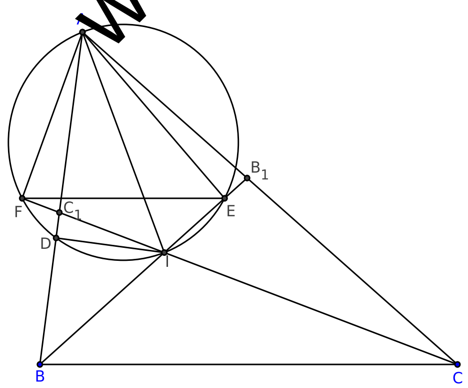
Solution: First let us count all the admissible triples having A_1A_2 as one of the sides. Having chosen A_1A_2, we cannot choose A_2A_3, A_3A_4, $A_{20}A_1$ nor $A_{19}A_{20}$. Thus we have to choose two sides separated by 2 sides among 15 sides A_4A_5, A_5A_6, …, $A_{18}A_{19}$. If A_4A_5 is one of them, the choice for the remaining side is only from 12 sides A_7A_8, A_8A_9, …, $A_{18}A_{19}$. If we choose A_5A_6 after A_1A_2, the choice for the third side is now only from A_8A_9, A_9A_{10}, …, $A_{18}A_{19}$ (11 sides). Thus the number of choices progressively decreases and finally for the side $A_{15}A_{16}$ there is only one choice, namely, $A_{18}A_{19}$. Hence the number of triples with A_1A_2 as one of the sides is

$$12 + 11 + 10 + \cdots + 1 = \frac{12 \times 13}{2} = 78.$$

Hence the number of triples then would be $(78 \times 20)/3 = 520$.

Remark: For an n-sided polygon, the number of such triples is $\dfrac{n(n-7)(n-8)}{6}$, for $n \geq 9$. We may check that for $n = 20$, this gives $(20 \times 13 \times 12)/6 = 520$.

5. Let ABC be a triangle and let BB_1, CC_1 be respectively the bisectors of $\angle B$, $\angle C$ with B_1 on AC and C_1 on AB. Let E, F be the feet of perpendiculars drawn from A onto BB_1, CC_1 respectively. Suppose D is the point at which the incircle of ABC touches AB. Prove that $AD = EF$.



Solution: Observe that $\angle ADI = \angle AFI = \angle AEI = 90°$. Hence A, F, D, I, E all lie on the circle with AI as diameter. We also know

$$\angle BIC = 90° + \frac{\angle A}{2} = \angle FIE.$$

This gives

$$\angle FAE = 180° - \left(90° + \frac{\angle A}{2}\right) = 90° - \frac{\angle A}{2}.$$

We also have $\angle AID = 90° - \dfrac{\angle A}{2}$. Thus $\angle FAE = \angle AID$. This shows the chords FE and AD subtend equal angles at the circumference of the same circle. Hence they have equal lengths, i.e., $FE = AD$.

6. Find all pairs (x, y) of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1.$$

Solution: Observe that

$$x^2 + y + x + y^2 + \frac{1}{2} = \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \geq 0.$$

This shows that $x^2 + y + x + y^2 \geq (-1/2)$. Hence we have

$$\begin{aligned} 1 = 16^{x^2+y} + 16^{x+y^2} &\geq 2\left(16^{x^2+y} \cdot 16^{x+y^2}\right)^{1/2}, \quad \text{(by AM-GM inequality)} \\ &= 2\left(16^{x^2+y+x+y^2}\right)^{1/2} \\ &\geq 2(16)^{-1/4} = 1. \end{aligned}$$

Thus equality holds every where. We conclude that

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 0.$$

This shows that $(x, y) = (-1/2, -1/2)$ is the only solution, as can easily be verified.

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