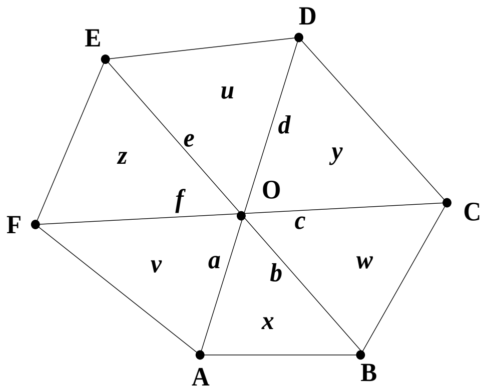
Regional Mathematical Olympiad-2010

Problems and Solutions

1. Let $ABCDEF$ be a convex hexagon in which the diagonals AD, BE, CF are concurrent at O. Suppose the area of traingle OAF is the geometric mean of those of OAB and OEF; and the area of triangle OBC is the geometric mean of those of OAB and OCD. Prove that the area of triangle OED is the geometric mean of those of OCD and OEF.



Solution: Let $OA = a$, $OB = b$, $OC = c$, $OD = d$, $OE = e$, $OF = f$, $[OAB] = x$, $[OCD] = y$, $[OEF] = z$, $[ODE] = u$, $[OFA] = v$ and $[OBC] = w$. We are given that $v^2 = zx$, $w^2 = xy$ and we have to prove that $u^2 = yz$.
Since $\angle AOB = \angle DOE$, we have

$$\frac{u}{x} = \frac{\frac{1}{2}de\sin\angle DOE}{\frac{1}{2}ab\sin\angle AOB} = \frac{de}{ab}.$$

Similarly, we obtain

$$\frac{v}{y} = \frac{fa}{cd}, \quad \frac{w}{z} = \frac{bc}{ef}.$$

Multiplying, these three equalities, we get $uvw = xyz$. Hence

$$x^2y^2z^2 = u^2v^2w^2 = u^2(zx)(xy).$$

This gives $u^2 = yz$, as desired.

2. Let $P_1(x) = ax^2 - bx - c$, $P_2(x) = bx^2 - cx - a$, $P_3(x) = cx^2 - ax - b$ be three quadratic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$.

Solution: We have three relations:

$$\begin{aligned} a\alpha^2 - b\alpha - c &= \lambda, \\ b\alpha^2 - c\alpha - a &= \lambda, \\ c\alpha^2 - a\alpha - b &= \lambda, \end{aligned}$$

where λ is the common value. Eliminating α^2 from these, taking these equations pair-wise,we get three relations:

$$(ca - b^2)\alpha - (bc - a^2) = \lambda(b - a), \quad (ab - c^2)\alpha - (ca - b^2) = \lambda(c - b),$$
$$(bc - a^2) - (ab - c^2) = \lambda(a - c).$$

Adding these three, we get

$$(ab + bc + ca - a^2 - b^2 - c^2)(\alpha - 1) = 0.$$

(Alternatively, multiplying above relations respectively by $b - c$, $c - a$ and $a - b$, and adding also leads to this.) Thus either $ab + bc + ca - a^2 - b^2 - c^2 = 0$ or $\alpha = 1$. In the first case

$$0 = ab + bc + ca - a^2 - b^2 - c^2 = \frac{1}{2}\Big((a - b)^2 + (b - c)^2 + (c - a)^2\Big)$$

shows that $a = b = c$. If $\alpha = 1$, then we obtain

$$a - b - c = b - c - a = c - a - b,$$

and once again we obtain $a = b = c$.

3. Find the number of 4-digit numbers(in base 10) having non-zero digits and which are divisible by 4 but not by 8.

Solution: We divide the even 4-digit numbers having non-zero digits into 4 classes: those ending in 2,4,6,8.

(A) Suppose a 4-digit number ends in 2. Then the second right digit must be odd in order to be divisible by 4. Thus the last 2 digits must be of the form 12, 32,52,72 or 92. If a number ends in 12, 52 or 92, then the previous digit must be even in order *not* to be divisible by 8 and we have 4 admissible even digits. Now the left most digit of such a 4-digit number can be any non-zero digit and there are 9 such ways, and we get $9 \times 4 \times 3 = 108$ such numbers. If a number ends in 32 or 72, then the previous digit must be odd in order *not* to be divisible by 8 and we have 5 admissible odd digits. Here again the left most digit of such a 4-digit number can be any non-zero digit and there are 9 such ways, and we get $9 \times 5 \times 2 = 90$ such numbers. Thus the number of 4-digit numbers having non-zero digits, ending in 2, divisible by 4 but not by 8 is $108 + 90 = 198$.

(B) If the number ends in 4, then the previous digit must be even for divisibility by 4. Thus the last two digits must be of the form 24, 44, 54, 84. If we take numbers ending with 24 and 64, then the previous digit must be odd for non-divisibility by 8 and the left most digit can be any non-zero digit. Here we get $9 \times 5 \times 2 = 90$ such numbers. If the last two digits are of the form 44 and 84, then previous digit must be even for non-divisibility by 8. And the left most digit can take 9 possible values. We thus get $9 \times 4 \times 2 = 72$ numbers. Thus the admissible numbers ending in 4 is $90 + 72 = 162$.

(C) If a number ends with 6, then the last two digits must be of the form 16,36,56,76,96. For numbers ending with 16, 56,76, the previous digit must be odd. For numbers ending with 36, 76, the previous digit must be even. Thus we get here $(9 \times 5 \times 3) + (9 \times 4 \times 2) = 135 + 72 = 207$ numbers.

(D) If a number ends with 8, then the last two digits must be of the form 28,48,68,88. For numbers ending with 28, 68, the previous digit must be even. For numbers ending with 48, 88, the previous digit must be odd. Thus we get $(9 \times 4 \times 2) + (9 \times 5 \times 2) = 72 + 90 = 162$ numbers.

Thus the number of 4-digit numbers, having non-zero digits, and divisible by 4 but not by 8 is

$$198 + 162 + 207 + 162 = 729.$$

Alternative Solution:. If we take any four consecutive even numbers and divide them by 8, we get remainders 0,2,4,6 in some order. Thus there is only one number of the form $8k + 4$ among them which is divisible by 4 but not by 8. Hence if we take four even consecutive numbers

$$1000a + 100b + 10c + 2, \quad 1000a + 100b + 10c + 4,$$
$$1000a + 100b + 10c + 6, \quad 1000a + 100b + 10c + 8,$$

there is exactly one among these four which is divisible by 4 but not by 8. Now we can divide the set of all 4-digit even numbers with non-zero digits into groups of 4 such consecutive even numbers with a, b, c nonzero. And in each group, there is exactly one number which is divisible by 4 but not by 8. The number of such groups is precisely equal to $9 \times 9 \times 9 = 729$, since we can vary $a, b.c$ in the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

4. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer.

Solution: Let x, y, z be three distinct positive integers satisfying the given conditions. We may assume that $x < y < z$. Thus we have three relations:

$$\frac{1}{y} + \frac{1}{z} = \frac{a}{x}, \quad \frac{1}{z} + \frac{1}{x} = \frac{b}{y}, \quad \frac{1}{x} + \frac{1}{y} = \frac{c}{z},$$

for some positive integers a, b, c. Thus

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a+1}{x} = \frac{b+1}{y} = \frac{c+1}{z} = r,$$

say. Since $x < y < z$, we observe that $a < b < c$. We also get

$$\frac{1}{x} = \frac{r}{a+1}, \quad \frac{1}{y} = \frac{r}{b+1}, \quad \frac{1}{z} = \frac{r}{c+1}.$$

Adding these, we obtain

$$r = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{r}{a+1} + \frac{r}{b+1} + \frac{r}{c+1},$$

or

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1. \qquad (1)$$

Using $a < b < c$, we get

$$1 = \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} < \frac{3}{a+1}.$$

Thus $a < 2$. We conclude that $a = 1$. Putting this in the relation (1), we get

$$\frac{1}{b+1} + \frac{1}{c+1} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence $b < c$ gives

$$\frac{1}{2} < \frac{2}{b+1}.$$

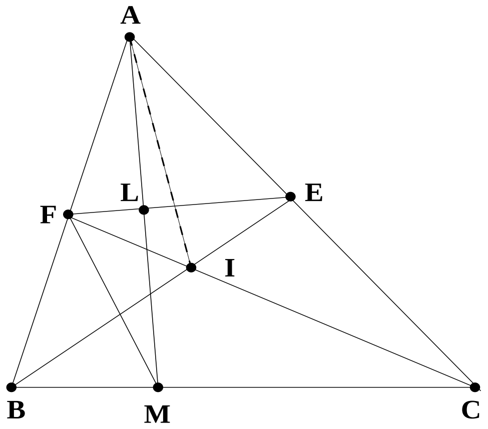
Thus $b + 1 < 4$ or $b < 3$. Since $b > a = 1$, we must have $b = 2$. This gives

$$\frac{1}{c+1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

or $c = 5$. Thus $x : y : z = a + 1 : b + 1 : c + 1 = 2 : 3 : 6$. Thus the required numbers with the least sum are 2,3,6.

Alternative Solution: We first observe that $(1, a, b)$ is not a solution whenever $1 < a < b$. Otherwise we should have $\frac{1}{a} + \frac{1}{b} = \frac{l}{1} = l$ for some integer l. Hence we obtain $\frac{a+b}{ab} = l$ showing that $a \big| b$ and $b \big| a$. Thus $a = b$ contradicting $a \neq b$. Thus the least number should be 2. It is easy to verify that $(2, 3, 4)$ and $(2, 3, 5)$ are not solutions and $(2, 3, 6)$ satisfies all the conditions.(We may observe $(2, 4, 5)$ is also not a solution.) Since $3 + 4 + 5 = 12 > 11 = 2 + 3 + 6$, it follows that $(2, 3, 6)$ has the required minimality.

5. Let ABC be a triangle in which $\angle A = 60°$. Let BE and CF be the bisectors of the angles $\angle B$ and $\angle C$ with E on AC and F on AB. Let M be the reflection of A in the line EF. Prove that M lies on BC.



Solution: Draw $AL \perp EF$ and extend it to meet AB in M. We show that $AL = LM$. First we show that A, F, I, E are concyclic. We have

$$\angle BIC = 90° + \frac{\angle A}{2} = 90° + 30° = 120°.$$

Hence $\angle FIE = \angle BIC = 120°$. Since $\angle A = 60°$, it follows that A, F, I, E are concyclic. Hence $\angle BEF = \angle IEF = \angle IAF = \angle A/2$. This gives

$$\angle AFE = \angle ABE + \angle BEF = \frac{\angle B}{2} + \frac{\angle A}{2}.$$

Since $\angle ALF = 90°$, we see that

$$\angle FAM = 90° - \angle AFE = 90° - \frac{\angle B}{2} - \frac{\angle A}{2} = \frac{\angle C}{2} = \angle FCM.$$

This implies that F, M, C, A are concyclic. It follows that

$$\angle FMA = \angle FCA = \frac{\angle C}{2} = \angle FAM.$$

Hence FMA is an isosceles triangle. But $FL \perp AM$. Hence L is the mid-point of AM or $AL = LM$.

6. For each integer $n \geq 1$, define $a_n = \left[\frac{n}{[\sqrt{n}]}\right]$, where $[x]$ denotes the largest integer not exceeding x, for any real number x. Find the number of all n in the set $\{1, 2, 3, \ldots, 2010\}$ for which $a_n > a_{n+1}$.

Solution: Let us examine the first few natural numbers: 1,2,3,4,5,6,7,8,9. Here we see that $a_n = 1, 2, 3, 2, 2, 3, 3, 4, 4, 3$. We observe that $a_n \leq a_{n+1}$ for all n except when $n+1$ is a square in which case $a_n > a_{n+1}$. We prove that this observation is valid in general. Consider the range

$$m^2, m^2+1, m^2+2, \ldots, m^2+m, m^2+m+1, \ldots, m^2+2m.$$

Let n take values in this range so that $n = m^2 + r$, where $0 \leq r \leq 2m$. Then we see that $[\sqrt{n}] = m$ and hence

$$\left[\frac{n}{[\sqrt{n}]}\right] = \left[\frac{m^2+r}{m}\right] = m + \left[\frac{r}{m}\right].$$

Thus a_n takes the values $\underbrace{m, m, m, \ldots, m}_{m \text{ times}}, \underbrace{m+1, m+1, m+1, \ldots, m+1}_{m \text{ times}}, m+2$, in this range. But when $n = (m+1)^2$, we see that $a_n = m+1$. This shows that $a_{n-1} > a_n$ whenever $n = (m+1)^2$. When we take n in the set $\{1, 2, 3, \ldots, 2010\}$, we see that the only squares are $1^2, 2^2, \ldots, 44^2$(since $44^2 = 1936$ and $45^2 = 2025$) and $n = (m+1)^2$ is possible for only 43 values of m. Thus $a_n > a_{n+1}$ for 43 values of n. (These are $2^2 - 1$, $3^2 - 1$, …, $44^2 - 1$.)

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