

## INMO-2010 Problems and Solutions

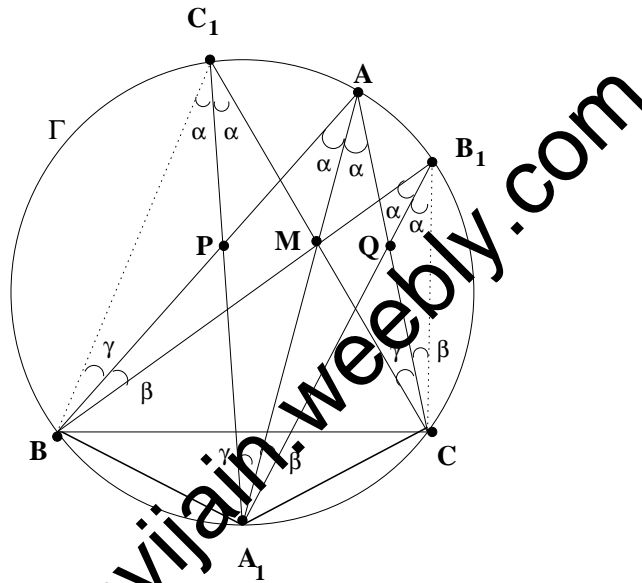
1. Let  $ABC$  be a triangle with circum-circle  $\Gamma$ . Let  $M$  be a point in the interior of triangle  $ABC$  which is also on the bisector of  $\angle A$ . Let  $AM, BM, CM$  meet  $\Gamma$  in  $A_1, B_1, C_1$  respectively. Suppose  $P$  is the point of intersection of  $A_1C_1$  with  $AB$ ; and  $Q$  is the point of intersection of  $A_1B_1$  with  $AC$ . Prove that  $PQ$  is parallel to  $BC$ .

**Solution:** Let  $A = 2\alpha$ . Then  $\angle A_1AC = \angle BAA_1 = \alpha$ . Thus

$$\angle A_1B_1C = \alpha = \angle BB_1A_1 = \angle A_1C_1C = \angle BC_1A_1.$$

We also have  $\angle B_1CQ = \angle AA_1B_1 = \beta$ , say. It follows that triangles  $MA_1B_1$  and  $QCB_1$  are similar and hence

$$\frac{QC}{MA_1} = \frac{B_1C}{B_1A_1}.$$



Similarly, triangles  $ACM$  and  $C_1A_1M$  are similar and we get

$$\frac{AC}{AM} = \frac{C_1A_1}{C_1M}.$$

Using the point  $P$ , we get similar ratios:

$$\frac{PB}{MA_1} = \frac{C_1B}{A_1C_1}, \quad \frac{AB}{AM} = \frac{A_1B_1}{MB_1}.$$

Thus,

$$\frac{QC}{PB} = \frac{A_1C_1 \cdot B_1C}{C_1B \cdot B_1A_1},$$

and

$$\begin{aligned} \frac{AC}{AB} &= \frac{MB_1 \cdot C_1A_1}{A_1B_1 \cdot C_1M} \\ &= \frac{MB_1}{C_1M} \frac{C_1A_1}{A_1B_1} = \frac{MB_1}{C_1M} \frac{C_1B \cdot QC}{PB \cdot B_1C}. \end{aligned}$$

However, triangles  $C_1BM$  and  $B_1CM$  are similar, which gives

$$\frac{B_1C}{C_1B} = \frac{MB_1}{MC_1}.$$

Putting this in the last expression, we get

$$\frac{AC}{AB} = \frac{QC}{PB}.$$

We conclude that  $PQ$  is parallel to  $BC$ .

2. Find all natural numbers  $n > 1$  such that  $n^2$  **does not** divide  $(n - 2)!$ .

**Solution:** Suppose  $n = pqr$ , where  $p < q$  are primes and  $r > 1$ . Then  $p \geq 2$ ,  $q \geq 3$  and  $r \geq 2$ , not necessarily a prime. Thus we have

$$\begin{aligned} n - 2 &\geq n - p = pqr - p \geq 5p > p, \\ n - 2 &\geq n - q = q(pr - 1) \geq 3q > q, \\ n - 2 &\geq n - pr = pr(q - 1) \geq 2pr > pr, \\ n - 2 &\geq n - qr = qr(p - 1) \geq qr. \end{aligned}$$

Observe that  $p, q, pr, qr$  are all distinct. Hence their product divides  $(n - 2)!$ . Thus  $n^2 = p^2q^2r^2$  divides  $(n - 2)!$  in this case. We conclude that either  $n = pq$  where  $p, q$  are distinct primes or  $n = p^k$  for some prime  $p$ .

**Case 1.** Suppose  $n = pq$  for some primes  $p, q$ , where  $2 < p < q$ . Then  $p \geq 3$  and  $q \geq 5$ . In this case

$$\begin{aligned} n - 2 &> n - p = p(q - 1) > 2p, \\ n - 2 &> n - q = q(p - 1) > 2q. \end{aligned}$$

Thus  $p, q, 2p, 2q$  are all distinct numbers in the set  $\{1, 2, 3, \dots, n - 2\}$ . We see that  $n^2 = p^2q^2$  divides  $(n - 2)!$ . We conclude that  $n = 2q$  for some prime  $q \geq 3$ . Note that  $n - 2 = 2q - 2 < 2q$  in this case so that  $n^2$  does not divide  $(n - 2)!$ .

**Case 2.** Suppose  $n = p^k$  for some prime  $p$ . We observe that  $p, 2p, 3p, \dots, (p^{k-1} - 1)p$  all lie in the set  $\{1, 2, 3, \dots, n - 2\}$ . If  $p^{k-1} - 1 \geq 2k$ , then there are at least  $2k$  multiples of  $p$  in the set  $\{1, 2, 3, \dots, n - 2\}$ . Hence  $n^2 = p^{2k}$  divides  $(n - 2)!$ . Thus  $p^{k-1} - 1 < 2k$ .

If  $k \geq 5$ , then  $p^{k-1} - 1 \geq 2k$ , which may be proved by an easy induction. Hence  $k \leq 4$ . If  $k = 1$ , we get  $n = p$ , a prime. If  $k = 2$ , then  $p - 1 < 4$  so that  $p = 2$  or  $3$ ; we get  $n = 2^2 = 4$  or  $n = 3^2 = 9$ . For  $k = 3$ , we have  $p^2 - 1 < 6$  giving  $p = 2$ ;  $n = 2^3 = 8$  in this case. Finally,  $k = 4$  gives  $p^3 - 1 < 8$ . Again  $p = 2$  and  $n = 2^4 = 16$ . However  $n^2 = 2^8$  divides  $14!$  and hence is not a solution.

Thus  $n = p, 2p$  for some prime  $p$  or  $n = 8, 9$ . It is easy to verify that these satisfy the conditions of the problem.

3. Find all non-zero real numbers  $x, y, z$  which satisfy the system of equations:

$$\begin{aligned} (x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) &= xyz, \\ (x^4 + x^2y^2 + y^4)(y^4 + y^2z^2 + z^4)(z^4 + z^2x^2 + x^4) &= x^3y^3z^3. \end{aligned}$$

**Solution:** Since  $xyz \neq 0$ , We can divide the second relation by the first. Observe that

$$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2),$$

holds for any  $x, y$ . Thus we get

$$(x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) = x^2y^2z^2.$$

However, for any real numbers  $x, y$ , we have

$$x^2 - xy + y^2 \geq |xy|.$$

Since  $x^2y^2z^2 = |xy| |yz| |zx|$ , we get

$$|xy| |yz| |zx| = (x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) \geq |xy| |yz| |zx|.$$

This is possible only if

$$x^2 - xy + y^2 = |xy|, \quad y^2 - yz + z^2 = |yz|, \quad z^2 - zx + x^2 = |zx|,$$

hold simultaneously. However  $|xy| = \pm xy$ . If  $x^2 - xy + y^2 = -xy$ , then  $x^2 + y^2 = 0$  giving  $x = y = 0$ . Since we are looking for nonzero  $x, y, z$ , we conclude that  $x^2 - xy + y^2 = xy$  which is same as  $x = y$ . Using the other two relations, we also get  $y = z$  and  $z = x$ . The first equation now gives  $27x^6 = x^3$ . This gives  $x^3 = 1/27$  (since  $x \neq 0$ ), or  $x = 1/3$ . We thus have  $x = y = z = 1/3$ . These also satisfy the second relation, as may be verified.

4. How many 6-tuples  $(a_1, a_2, a_3, a_4, a_5, a_6)$  are there such that each of  $a_1, a_2, a_3, a_4, a_5, a_6$  is from the set  $\{1, 2, 3, 4\}$  and the six expressions

$$a_j^2 - a_j a_{j+1} + a_{j+1}^2$$

for  $j = 1, 2, 3, 4, 5, 6$  (where  $a_7$  is to be taken as  $a_1$ ) are all equal to one another?

**Solution:** Without loss of generality, we may assume that  $a_1$  is the largest among  $a_1, a_2, a_3, a_4, a_5, a_6$ . Consider the relation

$$a_1^2 - a_1 a_2 + a_2^2 = a_2^2 - a_2 a_3 + a_3^2.$$

This leads to

$$(a_1 - a_2)(a_1 + a_3 - a_2) = 0.$$

Observe that  $a_1 \geq a_2$  and  $a_3 > 0$  together imply that the second factor on the left side is positive. Thus  $a_1 = a_3 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}$ . Using this and the relation

$$a_3^2 - a_3 a_4 + a_4^2 = a_4^2 - a_4 a_5 + a_5^2,$$

we conclude that  $a_3 = a_5$  as above. Thus we have

$$a_1 = a_3 = a_5 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}.$$

Let us consider the other relations. Using

$$a_2^2 - a_2 a_3 + a_3^2 = a_3^2 - a_3 a_4 + a_4^2,$$

we get  $a_2 = a_4$  or  $a_2 + a_4 = a_3 = a_1$ . Similarly, two more relations give either  $a_4 = a_6$  or  $a_4 + a_6 = a_5 = a_1$ ; and either  $a_6 = a_2$  or  $a_6 + a_2 = a_1$ . Let us give values to  $a_1$  and count the number of six-tuples in each case.

- (A) Suppose  $a_1 = 1$ . In this case all  $a_j$ 's are equal and we get only one six-tuple  $(1, 1, 1, 1, 1, 1)$ .
- (B) If  $a_1 = 2$ , we have  $a_3 = a_5 = 2$ . We observe that  $a_2 = a_4 = a_6 = 1$  or  $a_2 = a_4 = a_6 = 2$ . We get two more six-tuples:  $(2, 1, 2, 1, 2, 1)$ ,  $(2, 2, 2, 2, 2, 2)$ .
- (C) Taking  $a_1 = 3$ , we see that  $a_3 = a_5 = 3$ . In this case we get nine possibilities for  $(a_2, a_4, a_6)$ ;

$$(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1).$$

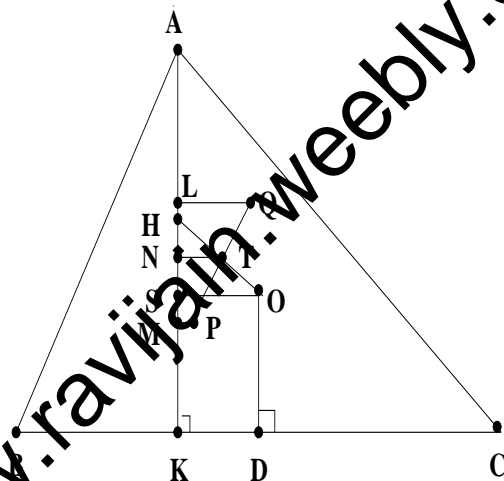
(D) In the case  $a_1 = 4$ , we have  $a_3 = a_5 = 4$  and

$$(a_2, a_4, a_6) = (2, 2, 2), (4, 4, 4), (1, 1, 1), (3, 3, 3), \\ (1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 3, 3), (3, 1, 3), (3, 3, 1).$$

Thus we get  $1 + 2 + 9 + 10 = 22$  solutions. Since  $(a_1, a_3, a_5)$  and  $(a_2, a_4, a_6)$  may be interchanged, we get 22 more six-tuples. However there are 4 common among these, namely,  $(1, 1, 1, 1, 1, 1)$ ,  $(2, 2, 2, 2, 2, 2)$ ,  $(3, 3, 3, 3, 3, 3)$  and  $(4, 4, 4, 4, 4, 4)$ . Hence the total number of six-tuples is  $22 + 22 - 4 = 40$ .

5. Let  $ABC$  be an acute-angled triangle with altitude  $AK$ . Let  $H$  be its ortho-centre and  $O$  be its circum-centre. Suppose  $KOH$  is an acute-angled triangle and  $P$  its circum-centre. Let  $Q$  be the reflection of  $P$  in the line  $HO$ . Show that  $Q$  lies on the line joining the mid-points of  $AB$  and  $AC$ .

**Solution:** Let  $D$  be the mid-point of  $BC$ ;  $M$  that of  $HK$ ; and  $T$  that of  $OH$ . Then  $PM$  is perpendicular to  $HK$  and  $PT$  is perpendicular to  $OH$ . Since  $Q$  is the reflection of  $P$  in  $HO$ , we observe that  $P, T, Q$  are collinear, and  $PT = TQ$ . Let  $QL, TN$  and  $OS$  be the perpendiculars drawn respectively from  $Q, T$  and  $O$  on to the altitude  $AK$ . (See the figure.)



We have  $LN = NM$ , since  $T$  is the mid-point of  $QP$ ;  $HN = NS$ , since  $T$  is the mid-point of  $OH$ ; and  $HM = MK$ , as  $P$  is the circum-centre of  $KHO$ . We obtain

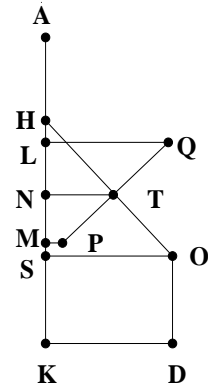
$$LH + HN = LN = NM = NS + SM,$$

which gives  $LH = SM$ . We know that  $AH = 2OD$ . Thus

$$AL = AH - LH = 2OD - LH = 2SK - SM = SK + (SK - SM) = SK + MK \\ = SK + HM = SK + HS + SM = SK + HS + LH = SK + LS = LK.$$

This shows that  $L$  is the mid-point of  $AK$  and hence lies on the line joining the midpoints of  $AB$  and  $AC$ . We observe that the line joining the mid-points of  $AB$  and  $AC$  is also perpendicular to  $AK$ . Since  $QL$  is perpendicular to  $AK$ , we conclude that  $Q$  also lies on the line joining the mid-points of  $AB$  and  $AC$ .

**Remark:** It may happen that  $H$  is above  $L$  as in the adjoining figure, but the result remains true here as well. We have  $HN = NS$ ,  $LN = NM$ , and  $HM = MK$  as earlier. Thus  $HN = HL + LN$  and  $NS = SM + NM$  give  $HL = SM$ . Now  $AL = AH + HL = 2OD + SM = 2SK + SM = SK + (SK + SM) = SK + MK = SK + HM = SK + HL + LM = SK + SM + LM = LK$ . The conclusion that  $Q$  lies on the line joining the mid-points of  $AB$  and  $AC$  follows as earlier.



6. Define a sequence  $\langle a_n \rangle_{n \geq 0}$  by  $a_0 = 0$ ,  $a_1 = 1$  and

$$a_n = 2a_{n-1} + a_{n-2},$$

for  $n \geq 2$ .

- (a) For every  $m > 0$  and  $0 \leq j \leq m$ , prove that  $2a_m$  divides  $a_{m+j} + (-1)^j a_{m-j}$ .
- (b) Suppose  $2^k$  divides  $n$  for some natural numbers  $n$  and  $k$ . Prove that  $2^k$  divides  $a_n$ .

**Solution:**

- (a) Consider  $f(j) = a_{m+j} + (-1)^j a_{m-j}$ ,  $0 \leq j \leq m$ , where  $m$  is a natural number. We observe that  $f(0) = 2a_m$  is divisible by  $2a_m$ . Similarly,

$$f(1) = a_{m+1} - a_{m-1} = 2a_m$$

is also divisible by  $2a_m$ . Assume that  $2a_m$  divides  $f(j)$  for all  $0 \leq j < l$ , where  $l \leq m$ . We prove that  $2a_m$  divides  $f(l)$ . Observe

$$\begin{aligned} f(l-1) &= a_{m+l-1} + (-1)^{l-1} a_{m-l+1}, \\ f(l-2) &= a_{m+l-2} + (-1)^{l-2} a_{m-l+2}. \end{aligned}$$

Thus we have

$$\begin{aligned} a_{m+l} &= a_{m+l-1} + a_{m+l-2} \\ &= 2f(l-1) - 2(-1)^{l-1} a_{m-l+1} + f(l-2) - (-1)^{l-2} a_{m-l+2} \\ &= 2f(l-1) + f(l-2) + (-1)^{l-1} (a_{m-l+2} - 2a_{m-l+1}) \\ &= 2f(l-1) + f(l-2) + (-1)^{l-1} a_{m-l}. \end{aligned}$$

This gives

$$f(l) = 2f(l-1) + f(l-2).$$

By induction hypothesis  $2a_m$  divides  $f(l-1)$  and  $f(l-2)$ . Hence  $2a_m$  divides  $f(l)$ . We conclude that  $2a_m$  divides  $f(j)$  for  $0 \leq j \leq m$ .

- (b) We see that  $f(m) = a_{2m}$ . Hence  $2a_m$  divides  $a_{2m}$  for all natural numbers  $m$ . Let  $n = 2^k l$  for some  $l \geq 1$ . Taking  $m = 2^{k-1} l$ , we see that  $2a_m$  divides  $a_n$ . Using an easy induction, we conclude that  $2^k a_l$  divides  $a_n$ . In particular  $2^k$  divides  $a_n$ .